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STRESS INTENSITY FACTORS DUE TO NON-ELASTIC STRAINS AND BODY FORCES FOR STEADY DYNAMIC CRACK EXTENSION IN AN ANISOTROPIC ELASTIC MATERIAL

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Abstract—Explicit solutions of the stress intensity factors due to non-elastic strains and body forces are obtained for a crack extending dynamically at steady state through an anisotropic elastic body. The examples of non-elastic strains include thermal expansion, phase transformation, initial strains, plastic strains, etc. Specific results are given for creeping materials, a uniform distribution of transformation strains, and discrete dislocations. It is shown that the stress intensity factors of tractions applied on the crack faces are independent of the crack velocity and the elastic constants.

1. INTRODUCTION

In this paper stress intensity factors due to non-elastic strains and body forces are derived for a dynamic crack in a linear anisotropic elastic body. The term "non-elastic strain" is equivalent to the stress-free transformation strain introduced by Eshelby (1957). However, non-elastic strain is adopted here in a broader context to denote such strains as thermal strain, transformation strain, plastic strain, and misfit strain. A characteristic feature of non-elastic strains is that they can all be expressed as the difference between a total strain and an elastic strain. The total strains are derived from continuous displacements while the elastic strains are related to the stresses through Hooke's law. As was illustrated by Eshelby (1957), the presence of such non-elastic strains in an elastic body causes a self-equilibrated internal stress or self-stress to occur. One may therefore regard non-elastic strains as sources of self-stress in an elastic body.

If a growing crack exists in a body containing non-elastic strains, the crack tip would be loaded by the self-stress due to the non-elastic strains in addition to the stress applied by the external loading. Under appropriate conditions regarding the distributions of non-elastic strains which will be discussed in the sequel, the self-stresses exhibit the universal crack-tip fields associated with a crack propagating through a linear elastic material. The crack-tip self-stress fields are characterized by a stress singularity of $r^{1/2}$, r being the radial distance from the moving crack tip and by the stress intensity factors. The stress intensity factors are functions of the non-elastic strains and are important parameters in describing the interactions between the non-elastic strains and the crack tip. For example, if the non-elastic strains are identified as the transformation strains due to phase transformation of zirconia particles in a ceramic matrix, the stress intensity factors represent the apparent toughness enhancement of the zirconia particles (Evans and Heuber, 1982; Budiansky *et al.*, 1983). In the case of plastic strains, the stress intensity factors characterize the stress relief effects by plastic flowing (Wu and Hart, 1987; Freund and Hutchinson, 1985).

In the aforementioned analyses, the derivations are either based on the application of the weight function method of Bueckner (1970) and Rice (1972), or by direct stress calculations. Furthermore, almost all the available results are applicable to quasi-static crack growth in an isotropic solid. The objective of the present study is to obtain general explicit expressions of the stress intensity factors due to non-elastic strains that are valid for dynamic crack extension through an anisotropic material. The stress intensity factors due to body forces are also considered since the derivation involved is essentially the same. The derivation will be carried out without solving the complete self-stresses due to the non-elastic

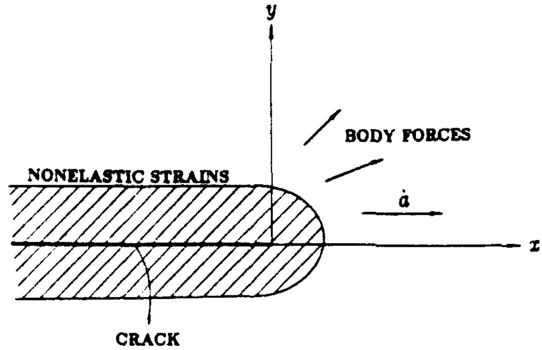

Fig. 1. A crack propagating dynamically at steady state in the presence of non-elastic strains and body forces.

strains and the body forces. Instead, the energy balance of the system containing the crack, the non-elastic strains, and the body forces is utilized to yield the solutions.

The plan of the paper is as follows. In Section 2, the underlying assumptions and the formulation of the problem concerned are given. The derivation of the stress intensity factors is shown in Section 3. Specific results are given in Section 4 for creeping materials, a uniform distribution of transformation strains, discrete dislocations and a point force acting on the crack face.

2. PROBLEM STATEMENT

In the following discussions, the summation convention over repeated indices is adopted unless otherwise noted. All indices range from 1 to 3.

In this study, it is assumed that the body forces are distributed near the tip of a dynamically extending crack in a region with dimensions much smaller than the crack length. For non-elastic strains, however, the assumption is slightly modified. The non-elastic strains are assumed to exist in a region extending along the crack flanks. The height of the zone of non-elastic strains is assumed to be small compared with the crack length. This assumption for the non-elastic strains stems from the concept of "small scale transformation" (Budiansky *et al.*, 1983) and "small scale creeping" (Wu and Hart, 1987). Under these conditions, a wake of non-zero non-elastic strains is left behind the advancing crack tip. The body forces and the non-elastic strains are further assumed to translate with the crack tip at steady state. Under such assumptions, the problem can be formulated as a semi-infinite crack extending dynamically at velocity $\dot{a}$ in an infinite body as shown in Fig. 1, where a moving coordinate system (x, y) is attached to the crack tip and the crack is assumed to coincide with the negative x-axis. The non-elastic strain, $\boldsymbol{\varepsilon}^{(n)}$, and the body force, $\mathbf{f}$, in the cracked body are functions of x and y such that with respect to an observer at the moving crack tip the distributions remain unchanged. The magnitude of the body forces approaches zero as $r \to \infty$. The magnitude of the non-elastic strains also approaches zero as $r \to \infty$ except behind the crack at $x \to -\infty$ where the non-elastic strains may be nonzero for finite values of y.

In the context of the theory for small deformation, the total strain $\boldsymbol{\varepsilon}$, is expressed as the sum of the non-elastic strain $\boldsymbol{\varepsilon}^{(n)}$ and the elastic strain $\mathbf{e}$. Namely

$$e_{ij} + \varepsilon_{ij}^{(n)} = \varepsilon_{ij} \tag{1}$$

where the total strain is given by

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

where $\mathbf{u}$ is the displacement and a comma denotes differentiation. Corresponding to the elastic deformation, a stress, $\boldsymbol{\sigma}$, is generated according to Hooke's law

$$\sigma_{ij} = C_{ijkl}e_{kl} \tag{3}$$

where **C** is the elasticity tensor. The stresses also satisfy the equilibrium equations

$$\sigma_{ij,j}+f_i = \rho\dot{a}^2\frac{\partial^2 u_i}{\partial x^2} \tag{4}$$

where ρ is the density, and the term on the right-hand side represents the inertial force. Equations (1)–(4) constitute the set of governing equations for the elasticity problem of determining the stresses and the displacements for *given* distributions of non-elastic strains and body forces. The set of equations must be supplemented by the traction-free conditions on the crack faces

$$\sigma_{i2} = 0 \tag{5}$$

and appropriate far-field conditions. Consistent with the assumptions stated previously, the stress remote from the crack tip is taken as (Freund, 1980; Budiansky *et al.*, 1983; Wu and Hart, 1987)

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(a)}+\boldsymbol{\sigma}^{(r)}, \quad \text{as} \quad r\to\infty \tag{6a}$$

where $\boldsymbol{\sigma}^{(a)}$ is the applied singular stress field for a dynamic crack in an elastic anisotropic material subject to the same external loading but neglecting the body forces and non-elastic strains and $\boldsymbol{\sigma}^{(r)}$ is the residual stress due to the non-vanishing non-elastic strains in the wake. The corresponding far-field displacement can be expressed as

$$\mathbf{u} = \mathbf{u}^{(a)}+\mathbf{u}^{(r)} \tag{6b}$$

where $\mathbf{u}^{(a)}$ is the displacement for $\boldsymbol{\sigma}^{(a)}$ and $\mathbf{u}^{(r)}$ is the displacement for $\boldsymbol{\sigma}^{(r)}$. The applied singular stress $\boldsymbol{\sigma}^{(a)}$ is given as (Wu, 1987)

$$\sigma_{ij}^{(a)} = \hat{\sigma}_{ijq}k_q^{(a)} \tag{7}$$

where $\hat{\boldsymbol{\sigma}}$ is given by

$$\hat{\sigma}_{i1q} = \frac{1}{\sqrt{(2\pi)}}\mathscr{R}\left[(\rho\dot{a}^2A_{i\alpha}-p_\alpha B_{i\alpha})B_{\alpha q}^{-1}\frac{1}{\sqrt{z_\alpha}}\right] \tag{8a}$$

$$\hat{\sigma}_{i2q} = \frac{1}{\sqrt{(2\pi)}}\mathscr{R}\left[B_{i\alpha}B_{\alpha q}^{-1}\frac{1}{\sqrt{z_\alpha}}\right]. \tag{8b}$$

In eqns (8a) and (8b) **A**, **B** are complex matrices and **p** is a complex vector determined by the elasticity constants; $z_\alpha = x+ip_\alpha y$, α varies from 1 to 3 and $\mathscr{R}$ denotes the real part. The procedure for determining **A**, **B**, and **p** from the elasticity constants can be found in Appendix A. The applied stress intensity factor $\mathbf{k}^{(a)}$ is determined for the actual geometry of the cracked body at a given external load in the absence of the body forces and the non-elastic strains. Note that since the applied stress is obtained by disregarding the non-elastic strains, the elastic strain $\mathbf{e}^{(a)}$ associated with the applied stress is the same as the total strain, i.e.

$$e_{ij}^{(a)} = \tfrac{1}{2}(u_{i,j}^{(a)}+u_{j,i}^{(a)}). \tag{8c}$$

The residual stress $\boldsymbol{\sigma}^{(r)}$ is determined by the following conditions:

$$\frac{\partial u_i^{(r)}}{\partial x} = 0 \tag{9a}$$

$$\sigma_{i2}^{(r)} = 0. \tag{9b}$$

Equation (9a) is imposed to insure the satisfaction of the steady-state conditions as $x \to -\infty$ for the displacements $\mathbf{u}^{(r)}$ associated with the residual stresses. The residual stresses consistent with the above conditions at $x \to -\infty$ can be shown to be

$$\sigma_{i1}^{(r)} = (C_{i1p2}T_{pj}^{-1}C_{j2kl} - C_{i1kl})\bar{\varepsilon}_{kl}^{(n)}(y) \tag{10}$$

where $T_{pj} = C_{p2j2}$ and $\mathbf{T}^{-1}$ is the inverse matrix of $\mathbf{T}$. $\bar{\varepsilon}^{(n)}(y)$ is the non-elastic strain as $x \to -\infty$. The derivation of eqn (10) is given in Appendix B. Inspection of eqns (1) and (4) reveals that the existence of the remote stress fields (6a) implies that the non-elastic strains and the body forces must vanish faster than $r^{-1/2}$ and $r^{-3/2}$, respectively, as $r \to \infty$ except $x \to -\infty$. As $x \to -\infty$, the non-elastic strains may be nonzero for finite values of y but must decay to zero faster than $|y|^{-1/2}$ as $y \to \pm\infty$.

Note that the stresses and the displacements of the problem outlined above are combinations of linear functions of the elastic stress intensity factor $\mathbf{k}^{(a)}$, the non-elastic strain $\boldsymbol{\varepsilon}^{(n)}$, and the body force, $\mathbf{f}$. The stresses and the displacements can thus be expressed as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(a)} + \boldsymbol{\sigma}^{(s)} \tag{11a}$$

$$\mathbf{u} = \mathbf{u}^{(a)} + \mathbf{u}^{(s)} \tag{11b}$$

where $\mathbf{u}^{(a)}$ and $\boldsymbol{\sigma}^{(a)}$ are the displacement and the stress due to $\mathbf{k}^{(a)}$ only and $\mathbf{u}^{(s)}$ and $\boldsymbol{\sigma}^{(s)}$ are the displacement and the stress induced by the non-elastic strains and the body forces. By definition, the stress $\boldsymbol{\sigma}^{(a)}$ is the stress in the absence of the non-elastic strains and the body forces and therefore is the same as the elastic singular field given by eqn (7). The displacement associated with the elastic singular stress field, $\mathbf{u}^{(a)}$, is given as (Wu, 1987)

$$u_i^{(a)} = \hat{u}_{iq}k_q^{(a)} \tag{12}$$

where

$$\hat{u}_{iq} = \sqrt{\left(\frac{2}{\pi}\right)}\mathscr{R}[A_{i\alpha}B_{\alpha q}^{-1}\sqrt{z_\alpha}]. \tag{13}$$

The problem one is concerned with here is that for which the stresses near the crack tip have the same form as eqn (7) except that the amplitude will in general be different, i.e.

$$\sigma_{ij} = \hat{\sigma}_{ijq}k_q \tag{14}$$

where $\mathbf{k}$ is the local stress intensity factor. The existence of the local crack-tip fields requires that the non-elastic strains and the body forces become unbounded more slowly than $r^{-1/2}$ and $r^{-3/2}$, respectively, as $r \to 0$. Equations (11a) and (14) imply that as $r \to 0$, the stress induced by the non-elastic strains and body forces $\boldsymbol{\sigma}^{(s)}$ also has the form of eqn (7) with the amplitude $\mathbf{k}^{(s)}$ given by

$$\mathbf{k}^{(s)} = \mathbf{k} - \mathbf{k}^{(a)}. \tag{15}$$

The objective is to determine the dynamic stress intensity factor $\mathbf{k}^{(s)}$ due to the non-elastic strains and the body forces. Of course the objective can be accomplished by going through the tedious process of deriving the complete stress fields from the governing equations and analyzing the asymptotic behavior of the stresses near the crack tip. One

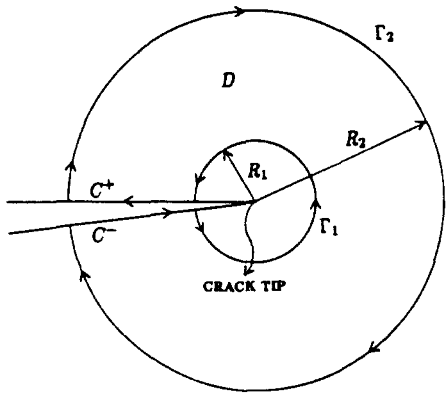

Fig. 2. The contour $\Gamma = \Gamma_1 \cup \Gamma_2 \cup C^+ \cup C^-$.

shall, however, adopt a simpler approach that allows the stress intensity factors to be obtained without actually solving the governing equations.

3. STRESS INTENSITY FACTORS DUE TO NON-ELASTIC STRAINS AND BODY FORCES

Consider a contour integral J' defined by

$$J' = \int_\Gamma \left[(W^e + T)\, dy - t_i \frac{\partial u_i}{\partial x}\, dl \right] \tag{16}$$

where W^e is the elastic strain energy density, T the kinetic energy density, and **t** the traction on Γ. The integral J' is an extended form of the path integral introduced for quasi-static crack growth in elastic–viscoplastic materials (Wu and Hart, 1987). A similar integral was proposed by Freund and Hutchinson (1985) by replacing the elastic strain energy density with the total strain energy density. Let $\Gamma = \Gamma_1 \cup \Gamma_2 \cup C^+ \cup C^-$ be the path as shown in Fig. 2, where Γ_1 and Γ_2 are circular paths with radii R_1 and R_2 and C^+ and C^- are the upper and the lower crack faces between R_1 and R_2. Applying the divergence theorem to eqn (16) and enforcing eqns (1) and (4) gives

$$J'_2 - J'_1 = \int_D \left(f_i \frac{\partial u_i}{\partial x} - \sigma_{ij} \frac{\partial \varepsilon_{ij}^{(n)}}{\partial x} \right) dA \tag{17}$$

where D is the region bounded by Γ and J'_1 and J'_2 are the values of J' evaluated over Γ_1 and Γ_2, respectively, in a counterclockwise sense. Integration over C^+ and C^- vanishes since both $dy = 0$ and $\mathbf{t} = \mathbf{0}$. Equation (17) states that for any solutions of the stress and the displacement, the difference between the values of J'_1 and J'_2 obtained by evaluating J' on any two non-intersecting contours Γ_1 and Γ_2, respectively, is equal to the work done by the non-elastic strain and the body force within region D.

Now take $R_1 \to 0$. On Γ_1 the stress fields are then given by eqn (14). The value of J'_1 can be shown to be equal to the local energy release rate given by (Wu, 1987)

$$J'_1 = \tfrac{1}{2} L_{ij} k_i k_j \tag{18}$$

where **L** is a symmetric and positive-definite matrix defined by

$$\mathbf{L} = -\mathscr{I}[\mathbf{A}\mathbf{B}^{-1}] \tag{19}$$

where $\mathscr{I}$ denotes the imaginary part.

On the other hand, let $R_2 \to \infty$ so that the stresses on Γ_2 are given by eqns (6). As shown in Appendix C, the value of J'_2 is determined as

$$J'_2 = \tfrac{1}{2}L_{ij}k_i^{(a)}k_j^{(a)} - H(\bar{\varepsilon}^{(n)}) \tag{20}$$

where $H(\bar{\varepsilon}^{(n)})$ is

$$H(\bar{\varepsilon}^{(n)}) = -\frac{1}{2}\int_{-\infty}^{\infty} \sigma_{i1}^{(r)}\bar{\varepsilon}_{i1}^{(n)}\,\mathrm{d}y. \tag{21}$$

Substituting eqns (18) and (21) into eqn (17) and rearranging the terms, one has

$$\tfrac{1}{2}L_{ij}k_ik_j + \int\left(f_i\frac{\partial u_i}{\partial x} - \sigma_{ij}\frac{\partial \varepsilon_{ij}^{(n)}}{\partial x}\right)\mathrm{d}A + H(\bar{\varepsilon}^{(n)}) = \tfrac{1}{2}L_{ij}k_i^{(a)}k_j^{(a)}. \tag{22}$$

The physical meaning of eqn (22) is that as the crack advances by an infinitesimal amount, the work done by the crack extension, the body forces, and the non-elastic strains represented by the first and the second terms, together with the residual energy in the wake given by the third term must be equal to the energy released by the external loading. Substitution of eqns (11a), (11b) and (15) into eqn (22) yields

$$\left[L_{pq}k_p^{(s)} - \int\left(\hat{\sigma}_{ijq}\frac{\partial \varepsilon_{ij}^{(n)}}{\partial x} - f_i\frac{\partial \hat{u}_{iq}}{\partial x}\right)\mathrm{d}A\right]k_q^{(a)}$$
$$+\tfrac{1}{2}L_{ij}k_i^{(s)}k_j^{(s)} + \int\left(f_i\frac{\partial u_i^{(s)}}{\partial x} - \sigma_{ij}^{(s)}\frac{\partial \varepsilon_{ij}^{(n)}}{\partial x}\right)\mathrm{d}A + H(\bar{\varepsilon}^{(n)}) = 0. \tag{23}$$

Since $\mathbf{k}^{(a)}$, $\boldsymbol{\varepsilon}^{(n)}$ and $\mathbf{f}$ are independent and arbitrary, it follows that

$$\tfrac{1}{2}L_{ij}k_i^{(s)}k_j^{(s)} + \int\left(f_i\frac{\partial u_i^{(s)}}{\partial x} - \sigma_{ij}^{(s)}\frac{\partial \varepsilon_{ij}^{(n)}}{\partial x}\right)\mathrm{d}A + H(\bar{\varepsilon}^{(n)}) = 0 \tag{24}$$

and

$$L_{pq}k_p^{(s)} = \int\left(\hat{\sigma}_{ijq}\frac{\partial \varepsilon_{ij}^{(n)}}{\partial x} - f_i\frac{\partial \hat{u}_{iq}}{\partial x}\right)\mathrm{d}A \tag{25}$$

or

$$k_p^{(s)} = L_{pq}^{-1}\int\left(\hat{\sigma}_{ijq}\frac{\partial \varepsilon_{ij}^{(n)}}{\partial x} - f_i\frac{\partial \hat{u}_{iq}}{\partial x}\right)\mathrm{d}A \tag{26}$$

where $\mathbf{L}^{-1}$ is the inverse matrix of $\mathbf{L}$. Equation (26) is the main result of this section. It represents the closed-form solutions to the problem of interest. Note that the introduction of the wake of non-elastic strains does not explicitly affect the forms of the solutions. Given an arbitrary distribution of non-elastic strains and/or body forces, the induced stress intensity factors can be computed by evaluating the integrals of eqn (26). In the following section, several applications of eqn (26) are given.

4. APPLICATIONS

4.1. *Creeping materials*

Because of the assumption of steady state, the non-elastic strain rates $\dot{\boldsymbol{\varepsilon}}^{(n)}$ are given by

$$\dot{\boldsymbol{\varepsilon}}^{(n)} = -\dot{a}\frac{\partial \boldsymbol{\varepsilon}^{(n)}}{\partial x}. \tag{27}$$

Thus

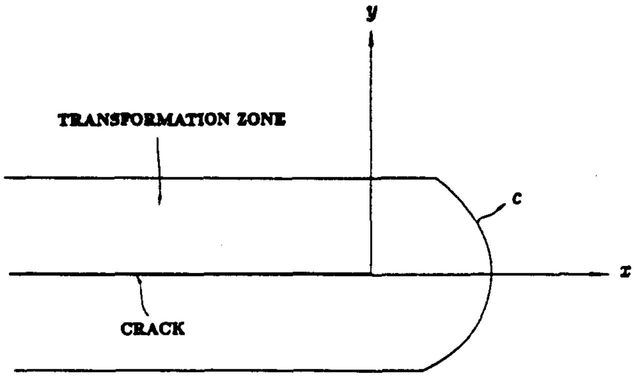

Fig. 3. Distribution of non-elastic strains simulating martensitic transformation in ceramics.

$$\frac{\partial \varepsilon^{(n)}}{\partial x} = -\frac{1}{\dot{a}}\dot{\varepsilon}^{(n)}. \tag{28}$$

The stress intensity factors due to the non-elastic flow are obtained by substituting eqn (28) into eqn (26)

$$k_p^{(s)} = -\frac{1}{\dot{a}} L_{pq}^{-1} \int \hat{\sigma}_{ijq} \dot{\varepsilon}_{ij}^{(n)} \, \mathrm{d}A. \tag{29}$$

Equation (29) is an extension of the result obtained for the quasi-static crack growth in isotropic materials (Wu and Hart, 1987). In the work of Wu and Hart (1987), they found that if the non-elastic strain rates are related to the stresses by a power law near the crack tip, the stresses are characterized by an $r^{1/2}$ singularity if the power is less than 3. Examination of eqn (29) shows that this conclusion also holds for the anisotropic dynamic case.

4.2. *Transformation toughening*

For many ceramics, a mechanism of toughness enhancement is by the stress-induced strain transformation of the contained particles (McMeeking and Evans, 1982). The strain transformation can be represented by a uniform distribution of non-elastic strains in the transformation zone shown in Fig. 3. The corresponding stress intensity factors are given by

$$k_p^{(s)} = -L_{pq}^{-1} \varepsilon_{ij}^{(n)} \int_c \hat{\sigma}_{ijq} \, \mathrm{d}y \tag{30}$$

where c is the contour of the transformation zone front. The values of $k_p^{(s)}$ are generally negative for positive transformation strains. The crack tip can therefore be considered as shielded by the surrounding transformation strains. Equation (30) indicates that the steady-state shielding effect depends only on the front of the transformation zone.

4.3. *Discrete dislocations*

A discrete dislocation of Burger's vector **b** located at **c** can be described by the following non-elastic strains on the slip plane $x \leqslant c_1$, $y = c_2$ (Mura, 1969):

$$\varepsilon^{(n)} = \frac{1}{2}\begin{pmatrix} 0 & b_1 & 0 \\ b_1 & 2b_2 & b_3 \\ 0 & b_3 & 0 \end{pmatrix} H(c_1 - x)\delta(y - c_2) \tag{31}$$

where H is the Heaviside function and δ the Dirac delta function. Differentiation of eqn (31) with respect to x gives

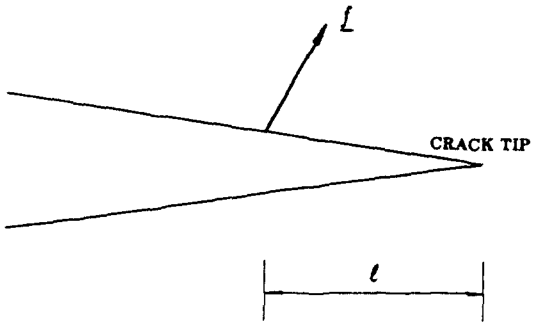

Fig. 4. A point load applied on the upper crack face.

$$\varepsilon^{(n)} = -\frac{1}{2}\begin{pmatrix} 0 & b_1 & 0 \\ b_1 & 2b_2 & b_3 \\ 0 & b_3 & 0 \end{pmatrix}\delta(c_1 - x)\delta(y - c_2). \tag{32}$$

Substituting eqn (32) into eqn (26) yields

$$\begin{aligned} k_p^{(s)} &= -L_{pq}^{-1}\hat{\sigma}_{i2q}b_i \\ &= -\frac{1}{\sqrt{(2\pi)}}L_{pq}^{-1}\mathscr{R}\left[B_{i\alpha}B_{\alpha q}^{-1}\frac{1}{\sqrt{c_\alpha}}\right]b_i \end{aligned} \tag{33}$$

where $c_\alpha = c_1 + p_\alpha c_2$. For stationary cracks, i.e. $\dot{a} \to 0$, eqn (33) reduces to the result obtained by Rice (1985) (the numerical factor of $(\pi)^{1/2}$ should be corrected to $\sqrt{\pi}$ in Rice's result).

4.4. *Point forces*

For a point force at $\mathbf{c}$, the induced stress intensity factors are immediately given by eqn (26) as

$$\begin{aligned} k_p^{(s)} &= -L_{pq}^{-1}\frac{\partial \hat{u}_{iq}}{\partial x}f_i \\ &= -\frac{1}{\sqrt{(2\pi)}}L_{pq}^{-1}\mathscr{R}\left[A_{i\alpha}B_{\alpha q}^{-1}\frac{1}{\sqrt{c_\alpha}}\right]f_i. \end{aligned} \tag{34}$$

An interesting case arises when a point load is applied to one of the crack faces, say, the upper face at $\mathbf{c} = (-l, 0)$ (Fig. 4). The corresponding $\mathbf{k}^{(s)}$ becomes

$$k_p^{(s)} = -\frac{1}{\sqrt{(2\pi)}}L_{pq}^{-1}\mathscr{I}[A_{i\alpha}B_{\alpha q}^{-1}]\frac{f_i}{\sqrt{l}}. \tag{35}$$

Invoking relationship (A5) shown in Appendix A

$$\mathbf{L} = -\mathscr{I}[\mathbf{A}\mathbf{B}^{-1}]$$

eqn (35) can be simplified to

$$k_p^{(s)} = \frac{f_p}{\sqrt{(2\pi l)}}. \tag{36}$$

This is a remarkable result that the stress intensity factors depend neither on the elastic constants nor on the crack velocity. Of course, this does not imply that the energy release rate G is not influenced. In fact, G is given by

$$G = \tfrac{1}{2} L_{pq} k_p^{(s)} k_q^{(s)}$$

$$= \frac{1}{4\pi l} L_{pq} f_p f_q \tag{37}$$

and **L** is both a function of the elasticity constants and the crack velocity.

5. CONCLUDING REMARKS

The stress intensity factors due to non-elastic strains and body forces have been derived for a dynamically propagating crack in an anisotropic elastic body. The solutions obtained can also be applied to the case of stationary cracks by taking the limit of the crack velocity $\dot{a}$ to zero. The solutions are good approximations if the non-elastic strains are present near the crack tip over a length scale in the direction perpendicular to the crack line much smaller than the crack length and the body dimensions. As many defects can be modeled by suitable distributions of non-elastic strains, eqn (26) can be used to investigate interactions between a stationary or a growing crack and various defects in the vicinity of the crack tip.

With regard to the tractions applied on the crack faces, an interesting result is found that the stress intensity factors are not affected by the elastic constants and the crack extension velocity. A similar conclusion was reached by Barnett and Asaro (1972) for a finite stationary crack embedded in an infinite body. The crack considered by Barnett and Asaro (1972) was subjected to self-equilibrating systems of tractions on the crack faces. On the other hand, for the case considered here, the tractions need not be self-equilibrating and can be applied to either one of the crack faces.

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APPENDIX A

The procedure for determining **p**, **A**, and **B** is briefly outlined here. More detailed discussions can be found in Wu (1987).

Let

$$C_{i1j1} \equiv Q_{ij}$$
$$C_{i1j2} \equiv R_{ij}$$
$$C_{i2j2} \equiv T_{ij}.$$

For simplicity, p will be used to denote any one of the components of $\mathbf{p}$. The values of p are determined by solving the following eigenvalue problem:

$$[\mathbf{Q}-\rho \dot{a}^2\mathbf{I}+(\mathbf{R}+\mathbf{R}^{\mathrm{T}})p+\mathbf{T}p^2]\mathbf{a} = 0 \tag{A1}$$

where $\mathbf{I}$ is the unity matrix, $\mathbf{R}^{\mathrm{T}}$ is the transpose of $\mathbf{R}$ and $\mathbf{a}$ is the eigenvector associated with p. The condition of non-trivial solutions of $\mathbf{a}$ gives

$$\det\,[\mathbf{Q}-\rho \dot{a}^2\mathbf{I}+(\mathbf{R}+\mathbf{R}^{\mathrm{T}})p+\mathbf{T}p^2] = 0. \tag{A2}$$

Equation (A2) is a sextic equation in p. It can be shown (Stroh, 1962) that for subsonic motion, the roots are complex. Since the coefficients of eqn (A2) are real, it follows that the roots of p must appear as two sets of complex numbers, one set being conjugate to the other. If one denotes p_α, $\alpha = 1, 2, 3$, as the eigenvalues for which the imaginary parts are positive, matrix $\mathbf{A}$ is given by

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3] \tag{A3}$$

where $\mathbf{a}_\alpha$ is the eigenvector corresponding to p_α. Matrix $\mathbf{B}$ is related to $\mathbf{A}$ by

$$B_{i\alpha} = (R_{ji}+p_\alpha T_{ij})A_{j\alpha} \quad \text{(no sum on } \alpha). \tag{A4}$$

Matrix $\mathbf{L}$ appearing in eqn (18) is given by

$$\mathbf{L} = -\mathscr{I}[\mathbf{A}\mathbf{B}^{-1}]. \tag{A5}$$

APPENDIX B

The derivation of eqn (10) for the residual stress $\boldsymbol{\sigma}^{(r)}$ is given here. Since from eqn (9b), $\sigma_{i2}^{(r)} = 0$, it is only necessary to determine $\sigma_{i1}^{(r)}$.

By expressing the elastic strain as the difference between the total strain and the non-elastic strain, eqn (3) becomes

$$\sigma_{ij} = C_{ijkl}(\varepsilon_{kl}-\varepsilon_{kl}^{(n)}). \tag{B1}$$

With eqn (2), eqn (B1) can be further written as

$$\sigma_{ij} = C_{ijkl}(u_{k,l}-\varepsilon_{kl}^{(n)}). \tag{B2}$$

As $x \to -\infty$, conditions (9a) hold and $\mathbf{u} \to \mathbf{u}^{(r)}$, $\boldsymbol{\varepsilon}^{(n)} \to \bar{\boldsymbol{\varepsilon}}^{(n)}$. Thus

$$\sigma_{ij}^{(r)} = C_{ijk2}u_{k,2}^{(r)} - C_{ijkl}\bar{\varepsilon}_{kl}^{(n)}. \tag{B3}$$

The traction-free conditions require

$$C_{i2k2}u_{k,2}^{(r)} - C_{i2kl}\bar{\varepsilon}_{kl}^{(n)} = 0. \tag{B4}$$

Introducing $T_{ik} = C_{i2k2}$, $u_{p,2}$ is determined as

$$u_{p,2}^{(r)} = T_{pj}^{-1}C_{j2kl}\bar{\varepsilon}_{kl}^{(n)}. \tag{B5}$$

Equation (10) is obtained by substituting eqn (B5) into eqn (B3).

APPENDIX C

From eqns (6), the stresses far removed from the crack tip are given by

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(a)} + \boldsymbol{\sigma}^{(r)}.$$

The corresponding elastic strain energy density function W^e can be expressed as

$$\begin{aligned} W^e &= \tfrac{1}{2}\sigma_{ij}e_{ij} \\ &= (\sigma_{ij}^{(a)}+\sigma_{ij}^{(r)})(e_{ij}^{(a)}+e_{ij}^{(r)}) \\ &= \tfrac{1}{2}\sigma_{ij}^{(a)}e_{ij}^{(a)} + \tfrac{1}{2}\sigma_{ij}^{(r)}e_{ij}^{(r)} + \sigma_{ij}^{(r)}e_{ij}^{(a)}. \end{aligned} \tag{C1}$$

Since the stress $\sigma^{(r)}$ satisfies eqns (9a) and (9b), eqn (C1) can be further simplified into the following form:

$$W^* = \tfrac{1}{2}\sigma_{ij}^{(a)}e_{ij}^{(a)} - \tfrac{1}{2}\sigma_{i1}^{(r)}\bar{\varepsilon}_{i1}^{(n)} + \sigma_{i1}^{(r)}\frac{\partial u_i^{(a)}}{\partial x}. \tag{C2}$$

In arriving at eqn (C2), relation (8c) has been utilized. The kinetic energy density T is given by

$$T = \tfrac{1}{2}\rho\frac{\partial u_i}{\partial x}\frac{\partial u_i}{\partial x}. \tag{C3}$$

With eqns (6b) and (9a), one has

$$\frac{\partial u_i}{\partial x} = \frac{\partial u_i^{(a)}}{\partial x}. \tag{C4}$$

Thus

$$T = \tfrac{1}{2}\rho\frac{\partial u_i^{(a)}}{\partial x}\frac{\partial u_i^{(a)}}{\partial x}. \tag{C5}$$

One also has

$$\begin{aligned} t_i\frac{\partial u_i}{\partial x} &= t_i\frac{\partial u_i^{(a)}}{\partial x} \\ &= (t_i^{(a)} + t_i^{(r)})\frac{\partial u_i^{(a)}}{\partial x} \\ &= t_i^{(a)}\frac{\partial u_i^{(a)}}{\partial x} + \sigma_{i1}^{(r)}n_1\frac{\partial u_i^{(a)}}{\partial x}. \end{aligned} \tag{C6}$$

With eqns (C2), (C5) and (C6), J'_2 is given by

$$J'_2 = \int_{\Gamma_2}\left[\left(\tfrac{1}{2}\sigma_{ij}^{(a)}e_{ij}^{(a)} + \tfrac{1}{2}\rho\frac{\partial u_i^{(a)}}{\partial x}\frac{\partial u_i^{(a)}}{\partial x}\right)\mathrm{d}y - t_i^{(a)}\frac{\partial u_i^{(a)}}{\partial x}\,\mathrm{d}l\right] + \int_{-\infty}^{\infty}\tfrac{1}{2}\sigma_{i1}^{(r)}\bar{\varepsilon}_{i1}^{(n)}\,\mathrm{d}y + \int_{\Gamma_2}\left(\sigma_{i1}^{(r)}\frac{\partial u_i^{(a)}}{\partial x}\,\mathrm{d}y - \sigma_{i1}^{(r)}n_1\frac{\partial u_i^{(a)}}{\partial x}\,\mathrm{d}l\right). \tag{C7}$$

The first integral on the right-hand side of eqn (C7) is the same as eqn (18) except that $\mathbf{k}$ is now replaced by $\mathbf{k}^{(a)}$, i.e.

$$\int_{\Gamma_2}\left[\left(\tfrac{1}{2}\sigma_{ij}^{(a)}e_{ij}^{(a)} + \tfrac{1}{2}\rho\frac{\partial u_i^{(a)}}{\partial x}\frac{\partial u_i^{(a)}}{\partial x}\right)\mathrm{d}y - t_i^{(a)}\frac{\partial u_i^{(a)}}{\partial x}\,\mathrm{d}l\right] = \tfrac{1}{2}L_{ij}k_i^{(a)}k_j^{(a)}. \tag{C8}$$

The second integral is defined as

$$-\frac{1}{2}\int_{-\infty}^{\infty}\sigma_{i1}^{(r)}\bar{\varepsilon}_{i1}^{(n)}\,\mathrm{d}y = H(\bar{\varepsilon}^{(n)}). \tag{C9}$$

The last integral vanishes as $n_1\,\mathrm{d}l = \mathrm{d}y$. The final form for J'_2 is thus given by eqn (20).